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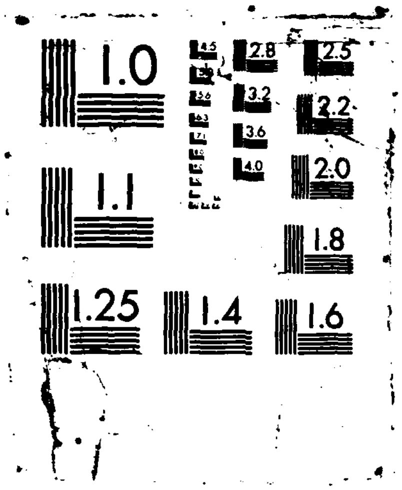
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APPROXIMATIONS AND OPTIMAL CONTROL FOR THE PATHWISE AVERAGE COST PER UNIT TIME AND DISCOUNTED PROBLEMS FOR WIDEBAND NOISE DRIVEN SYSTEMS

by
Harold J. Kushner

January 1988 LCDS/CCS #88-3

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ABSTRACT

We consider the average cost per unit time problem for wide bandwidth noise driven control systems, where the average cost is in the pathwise sense; no expectations are used. Let t = time of control and BW = bandwidth. For our class of processes, we prove various uniformity properties for the convergence of the pathwise average costs as $t \to \infty$, BW $\to \infty$. Let $u^{\delta}(\cdot)$ be a smooth δ-optimal control for the limit controlled diffusion (the limit as BW $\to \infty$) for the (mean) average cost per unit time problem. We show that for large enough t and BW, $u^{\delta}(\cdot)$ is 2δ-optimal (with a probability arbitrarily close to 1) for the pathwise wide bandwidth problem. This uniformity is important in applications, for we often have only one long sequence to control, and the expectation is inappropriate. Also, otherwise, as BW $\to \infty$, it might take longer and longer to well approximate the limit pathwise average cost. Applications to related 'pathwise average' problems are given: the convergence of the average pathwise errors for an 'approximate' non-linear filter with wide bandwidth observation and system driving noise, and the convergence and accuracy of Monte Carlo calculations of Liapunov exponents for wide bandwidth noise driven systems (as BW $\to \infty$) via average cost/unit time methods. It is also shown for the discounted cost problem that the optimum pathwise costs converge to the minimum average cost per unit time as both the discount factor goes to zero, and BW $\to \infty$.

Key words: pathwise average cost per unit time, ergodic control, approximations of ergodic control, wide band noise driven systems, approximate non-linear filtering, Liapunov exponents, discounted cost.

1. Introduction

Average cost per unit time (over an infinite time horizon) optimal control problems for diffusion and other Markov models have been dealt with in various ways ~~as in, e.g., [1], [2], [3]~~. We treat such a problem for 'wideband noise driven' and related systems, which are 'close' to a diffusion, and when the average is in the pathwise but not necessarily in the mean value sense. The general method works for many other classes of processes which are suitably approximated by an appropriate controlled Markov process. As pointed out below and in Sections 4 and 5, the results have applications to many other problems where pathwise averages are important, and the noises are 'wide band'. E.g., in Section 5, we treat the problem where both BW $\to \infty$ and discount factor $\to 0$.

Let the diffusion model be given in the relaxed control form (1.1), where $\bar{b}(\cdot,\cdot)$ and $\sigma(\cdot)$ are continuous (other conditions will be listed below) and $m_t(\cdot)$ is an admissible relaxed control [1], [3], [4], over a compact control value space U. The relaxed control might be of the feedback form. The precise definition is in the Appendix. We note here that $m_t(\cdot)$ is a measure over the Borel sets of U.

$$(1.1) \qquad dx = \int \bar{b}(x,\alpha)m_t(d\alpha)dt + \sigma(x)dw.$$

In [1], relaxed controls were used to get nearly optimal controls for several 'wideband' noise driven systems, and in [3], they were cleverly used to get an 'occupation measure' for the state-control pair which ultimately allowed the authors to demonstrate the existence of an optimal stationary control. These advantages also occur for the particular problems to be described below. In [1], [2], the cost of concern was ([2] did not use relaxed controls)

$$\overline{\lim_T} \frac{1}{T} \int_0^T Ek(x(t),\alpha)m_t(d\alpha) \equiv \bar{\gamma}(m), \tag{1.2}$$

for a bounded continuous $k(\cdot)$.

In practice, of course, one does not have a process which is a diffusion, and it is of considerable interest to consider wide bandwidth noise driven systems of the form

$$\dot{x}^\epsilon = \int b(x^\epsilon,\alpha)m_t(d\alpha) + F_\epsilon(x^\epsilon,\xi^\epsilon) \tag{1.3}$$

where $\xi^\epsilon(\cdot)$ is the wide bandwidth noise. We use the scaling $\xi^\epsilon(t) = \xi(t/\epsilon)$ for an appropriate 'mixing' process $\xi(\cdot)$ owing to its convenience in simplifying the details. But it should be clear that the method is of fairly general applicability. Reference [1] dealt with a system of type (1.3) (with weak limit of type (1.1)) and cost of the form (1.2). It was shown, under the conditions there that for any $\delta > 0$, a smooth δ-optimal control u^δ for (1.1), (1.2) was also 'nearly' optimal for (1.3) and (1.4), for small ϵ.

$$\overline{\lim_T} \frac{1}{T} \int_0^T Ek(x^\epsilon(t),\alpha)m_t(d\alpha) = \bar{\gamma}^\epsilon(m) \tag{1.4}$$

i.e., $\underline{\lim}_{\epsilon}\, \bar{\gamma}^\epsilon(m^\epsilon) \geq \lim_{\epsilon} \bar{\gamma}^\epsilon(u^\delta) - \delta$ for any sequence m^ϵ.
Such results are helpful in justifying the use of the ideal limit process (1.1) for use in control theory.

In [3], Borkar and Ghosh showed the existence of an optimal feedback control for the diffusion model (under this control the diffusion could be taken to be stationary) and cost function (1.2), but <u>with the</u> E <u>deleted</u> -- a pathwise result. This paper is devoted to a related problem for the model (1.3). Define

(1.5) $$\gamma_T(m) = \frac{1}{T}\int_0^T k(x(s),\alpha)m_s(d\alpha), \quad \gamma(m) = \overline{\lim_T}\ \gamma_T(m),$$

(1.6) $$\gamma_T^\epsilon(m) = \frac{1}{T}\int_0^T k(x^\epsilon(s),\alpha)m_s(d\alpha).$$

If $m(\cdot)$ is equivalent to a classical control function $u(\cdot)$, we write u in lieu of m in $\gamma_T(m)$, etc. The 'pathwise' convergence result in [3] is of particular importance in applications, since one often has a single long realization, and then the expectation is not appropriate in the cost function. The results in [3] (under their conditions) give the existence of a feedback relaxed control $\bar{m}(\cdot)$ such that $\gamma_T(\bar{m}) \to \gamma = \inf_m \overline{\lim_T}\ \gamma_T(m)$ w.p.1.

In our problem here, owing to the wideband noise and the appearance of the two parameters ϵ and T, w.p.1 type convergence results are usually either meaningless or impossible to obtain. Typically, in an application one has a particular process with a given wide bandwidth driving force. One is interested in knowing how well good controls for the 'limit' problem do on the actual physical problem. The wide bandwidth driving term is imbedded into a sequence for purposes of getting an approximation result, and w.p.1 type results might make little sense.

Let $u^\delta(\cdot)$ denote a 'nice' δ-optimal classical control ('nice' is defined in the next section) for model (1.1) and cost function (1.4). Then we wish to show (1.8a) and (1.8b):

(1.8a) $$\gamma_T^\epsilon(u^\delta) \xrightarrow{P} \bar{\gamma}(u^\delta), \quad \text{as } \epsilon \to 0,\ T \to \infty,$$

(1.8b) $$\underline{\lim}_{\epsilon,T}\ P\{\gamma_T^\epsilon(m^\epsilon) \geq \bar{\gamma}(u^\delta) - \delta\} = 1$$

for any sequence of admissible relaxed controls $m^\epsilon(\cdot)$. Since the time derivative of $\gamma_T^\epsilon(m)$ is $O(1/T)$ uniformly in ϵ, m, ω, the convergence is somewhat stronger than indicated by (1.8). Eqn. (1.8b) implies a type of uniformity of convergence, since the way that $\epsilon \to 0$ and $T \to \infty$ is not important. Were this 'uniformity' not the case, it would be possible that as $\epsilon \to 0$, a larger and larger T is needed in order to closely approximate the limit value. In that case, the white noise limit (1.1) would not be useful for predictive or control purposes, when the true model is (1.3).

In Section 2, we list several assumptions and prove (1.8). In order to simplify the development, the technique of perturbed test functions from [5] is used. To facilitate the calculations, some of the conditions will be adapted from those used in that reference -- but many useful generalizations should be clear. In Section 3, we redevelop the result of Section 2, using a 'first order perturbed test function' method, with less smoothness required on the functions and less mixing required on the noise but more details required in the proof. Some extensions are discussed in Section 4. The ideas of 'pathwise uniform' convergence of a sample average cost per unit time has many other applications. For example in the Monte Carlo evaluation of Liapunov exponents with wide bandwidth noise coefficients for linear systems [6]. The formula for these exponents is of the form of an average cost per unit time. For this problem, it is shown in Section 4 that the Monte Carlo evaluated pathwise average cost per unit time converges (as $\epsilon \to 0$, $T \to \infty$) to the same limit that one would obtain were the actual limit diffusion used for the evaluation. The limit depends only on the correlation function of the noise $\xi^\epsilon(\cdot)$. Such a result is essential for the Monte Carlo method to be useful, and for the Liapunov exponents of the limit system to be meaningful indicators of the behavior of the actual (wide bandwidth noise driven) physical system.

An extension to a problem of average pathwise error per unit time for an 'approximate' non-linear filter for a system with wide bandwidth driving and observation noise is also discussed in Section 4.

In Section 5, we treat extensions to the discounted cost case. Define the pathwise discounted cost

$$V_\beta^\epsilon(m) = \beta \int_0^\infty e^{-\beta s} \int k(x^\epsilon(s),\alpha)m_s(d\alpha)ds,$$

and let $m^\epsilon(\cdot)$ be a sequence of δ_1-optimal controls. We show that

(1.9a) $$V_\beta^\epsilon(u^\delta) \xrightarrow{P} \bar{\gamma}(u^\delta), \text{ as } \beta \to 0, \ \epsilon \to 0,$$

(1.9b) $$\underline{\lim}_{\epsilon,\beta} P\{V_\beta^\epsilon(m^\epsilon) \geq \bar{\gamma}(u^\delta) - \delta\} = 1.$$

The uniformity result is important, since we would not want the speed with which $\beta \to 0$ to depend on the bandwidth – in order to get the proper approximation. The sense in which $m^\epsilon(\cdot)$ is δ_1-optimal is left purposely vague – since (1.9) holds for any $\{m^\epsilon(\cdot)\}$, under the conditions below. Thus for small ϵ,β, $u^\delta(\cdot)$ is always nearly optimal. There also are extensions to impulsive and singular control problems.

2. A Basic Convergence Theorem

For convenience in this section, we use the assumptions of [5, Chapter 4.6], with appropriate modification for the relaxed controls. The system (1.3) will take the form

$$\dot{x}^{\epsilon} = \int \bar{G}(x^{\epsilon},\alpha)m_t(d\alpha) + G_0(x^{\epsilon},\xi^{\epsilon}) + F(x^{\epsilon},\xi^{\epsilon})/\epsilon. \tag{2.1}$$

(2.1) is a common way of getting a wide bandwidth noise driven system [5,13,14]. Other forms for $F(x,\xi)/\epsilon$ can be used. See, e.g., the examples in [5] where the use of perturbed test functions for weak convergence is illustrated. We use either bounded noise or Gaussian noise. For the first case (A2.1) - (A2.6) are used. The second case is covered by (A2.10). Let E_t^{ϵ} denote the expectation, conditioned on $\xi^{\epsilon}(s)$, $s \le t$, and E_t the expectation conditioned on $\xi(s)$, $s \le t$.

A2.1. $\bar{G}(\cdot)$, $F(\cdot,\cdot)$, $G_0(\cdot,\cdot)$, $F_x(\cdot,\cdot)$ <u>are continuous in</u> (x,ξ). $G_{0,x}(\cdot,\xi)$ <u>is continuous in</u> x <u>for each</u> ξ <u>and is bounded</u>. $\xi(\cdot)$ <u>is bounded, right continuous and</u> $EG_0(x,\xi) = EF(x,\xi) = 0$.

A2.2. $F_{xx}(\cdot,\xi)$ <u>is continuous for each</u> ξ, <u>and is bounded</u>.

A2.3. <u>Let</u> $V(x,\xi)$ <u>denote either</u> $\epsilon G_0(x,\xi)$, $G_x(x,\xi)$, $F(x,\xi)$ <u>or</u> $F_x(x,\xi)$. <u>Then for compact</u> Q,

$$\epsilon \sup_{x\in Q}\left|\int_{t/\epsilon^2}^{\infty} E_t^{\epsilon} V(x,\xi(s))ds\right| \xrightarrow{\epsilon} 0$$

<u>in the mean square sense, uniformly in</u> t.

Let F_i denote the ith component of F.

A2.4. <u>There are continuous</u> $\bar{F}_i(\cdot)$, $\bar{a}(\cdot)$ <u>such that</u>

$$\int_t^\infty EF'_{i,x}(x,\xi(s))F(x,\xi(t))ds \to \bar{F}_i(x),$$

$$\int_t^\infty EF_i(x,\xi(s))F_j(x,\xi(t))ds \to \tilde{a}_{ij}(x)/2,$$

as $t \to \infty$, and the convergence is uniform in any bounded x-set.

Define $a(x) = \frac{1}{2}[\tilde{a}(x) + \tilde{a}'(x)]$.

A2.5. For each compact set Q,

$$\sup_{x\in Q} \epsilon \left| \int_{t/\epsilon^2}^\infty d\tau \int_\tau^\infty ds[E_t^\epsilon F'_{i,x}(x,\xi(s))F(x,\xi(\tau)) - EF'_{i,x}(x,\xi(s))F(x,\xi(\tau))] \right| \to 0$$

$$\sup_{x\in Q} \epsilon \left| \int_{t/\epsilon^s}^\infty d\tau \int_\tau^\infty ds[E_t^\epsilon F(x,\xi(s))F'(x,\xi(\tau)) - EF(x,\xi(s))F'(x,\xi(\tau))] \right| \to 0$$

in the mean square sense as $\epsilon \to 0$, uniformly in t. Similarly, when the bracketed terms are replaced by their x-gradients.

Remark. (A2.4) is just a condition on the rate of convergence of an expectation to a 'stationary' value as $t \to \infty$. (A2.3) and (A2.5) are just conditions on the rate of convergence of a conditional expectation to an expectation as the 'time difference' goes to infinity. They are easily shown to be satisfied under appropriate mixing conditions on $\xi(\cdot)$ [7, Chapter 4]. They are similar to conditions used in [13,14] for weak convergence of a sequence of Markov processes.

Define $\bar{b}(x,\alpha) = \bar{G}(x,\alpha) + \bar{F}(x)$ and the operators A^m (when m is a feedback relaxed control m_x; see the Appendix for the definition) and A^α and A^u as follows:

$$A^{\alpha}f(x) = f_x'(x)\bar{b}(x,\alpha) + \frac{1}{2}\sum_{i,j} a_{ij}(x)f_{x_ix_j}(x),$$

$$A^m f(x) = \int A^{\alpha}f(x)m_x(d\alpha),$$

and for A^u, we replace the α in the definition of A^{α} by the classical control function $u(\cdot)$.

A2.6. The martingale problem for operator A^m has a unique solution for each relaxed admissible feedback control $m_x(\cdot)$, and each initial condition. The process is a Feller process. The solution of (2.1) is unique in the weak sense for each $\epsilon > 0$.

Remark. The uniqueness and existence is guaranteed if the operator A^m is that for the system

$$dx = \tilde{b}(x)dt + \begin{bmatrix} \int \hat{b}(x,\alpha)m_x(d\alpha)dt \\ 0 \end{bmatrix} + \begin{bmatrix} \sigma(x)dw \\ 0 \end{bmatrix}, \tag{2.2}$$

where $\sigma\sigma' \geq \delta I$ for all x and some $\delta > 0$, $\breve{b}(\cdot)$ and $\sigma(\cdot)$ are Lipschitz continuous and $\hat{b}(\cdot,\cdot)$ is merely bounded and Borel measurable and the dimensions of (the vector) $\hat{b}$ and (square matrix) $\sigma\sigma'$ are equal.

Let $\mathcal{M}$ denote the space of probability measures on the Borel sets of $R^r \times U$, with the 'weak compact' topology where $P_n \to P$ iff $\int f(x,\alpha)P_n(dxd\alpha) \to \int f(x,\alpha)P(dxd\alpha)$ for each continuous function $f(\cdot)$ with compact support. For an admissible relaxed control for (2.1) and (1.1), resp., define the (occupation) measure valued random variables $P_T^{m,\epsilon}(\cdot)$ and $P_T^m(\cdot)$ by, resp.,

$$P_T^{m,\epsilon}(B \times C) = \frac{1}{T}\int_0^T I_{\{x^\epsilon(t)\in B\}} m_t(C)dt,$$

$$P_T^m(B \times C) = \frac{1}{T}\int_0^T I_{\{x(t)\in B\}} m_t(C)dt.$$

We sometimes write $m^\epsilon(\cdot)$, if the model is (2.1). If the relaxed control for (1.1) is of the feedback form (m_x or $u(x)$) , then we use the modification

$$P_T^m(B) = \frac{1}{T}\int_0^T I_{\{x(t)\in B\}} dt$$

(or with u replacing m), and similarly define $P_T^{m,\epsilon}(B)$, $P_T^{u,\epsilon}(B)$ for feedback $m(\cdot)$ and $u(\cdot)$.

Let $m^\epsilon(\cdot)$ be δ_1-optimal (in any sense) and let $u^\delta(\cdot)$ be defined by (A2.8).

A2.7. <u>The set of random variables</u> $\{x^\epsilon(t), \epsilon > 0, t < \infty\}$ <u>is tight</u>.

Remark. The tightness in (A2.7) implies the tightness of the set of $\mathcal{M}$ valued random variables $\{P_T^{m,\epsilon}(\cdot), \epsilon > 0, T < \infty, u^\delta$ or above $m_t^\epsilon(\cdot)\}$. Under a stability condition on the limit equation (1.1) in the absence of control, and some other conditions, the tightness can be proved by a 'perturbed Liapunov function' method [5]. Of course, if the state space is compact, as for the 'Liapunov exponent' problem in Section 4, then (A2.7) always holds. In lieu of a 'universal stability condition', a condition on the minimum (over the control values) magnitude of the cost $k(\cdot)$ as $|x| \to \infty$ was used in [3] (for the model (1.1)) to get that an optimal control for that model is 'stabilizing'. Perhaps a similar idea can be used here. But this point won't be pursued.

A2.8. <u>For each</u> $\delta > 0$, <u>there is a continuous</u> δ<u>-optimal control for</u> (1.1) <u>and</u> (1.2), <u>for which the martingale problem has a unique solution for each initial</u>

condition. The solution is a Feller process and there is a unique invariant measure $\mu(u^\delta, \cdot)$. [u^δ is δ-optimal in the sense that $\bar{\gamma}(u^\delta) \leq \bar{\gamma}(m_x) + \delta$ for the stationary initial condition for any feedback relaxed control m_x for which there is a stationary solution to the associated martingale problem.]

A.2.9. $k(\cdot)$ is bounded and continuous.

Remark. The existence of such smooth δ-optimal controls is dealt with in [7]. It will exist under an appropriate stability condition on the uncontrolled (1.1), and either non-degeneracy of (1.1) or for a system of the form (2.2) [7]. It turns out that $\gamma(u^\gamma) = \bar{\gamma}(u^\gamma)$ w.p.1 (this follows from the method of proof of Theorem 1 below, or from the method in [3], under the conditions there).

A2.10. (Gaussian case). $\xi(\cdot)$ is a stable Gauss-Markov process with a stationary transition function and let $F(x,\xi) = F(x)\xi$, $G_0(x,\xi) = G_0(x)\xi$, where $\bar{G}$, G_0, and F satisfy the (in x) smoothness in (A2.1) - (A2.2). Define $\bar{F}(\cdot)$ and $a(\cdot)$ as in (A2.4). [Note: the other parts of (A2.3) - (A2.5)) all hold.]

Theorem 1. Assume either (A2.1) to (A2.9) or (A2.6) to (A2.10). Then (1.8a) and (1.8b) hold.

Proof. We do the 'Gaussian' case only. The other case is treated in essentially the same way. Let $\mathcal{F}$ be a (countable) measure determing set of bounded continuous functions which have continuous second partial derivation, and are constant for large $|x|$. Let $m_t^\epsilon(\cdot)$ be the relaxed control in (A2.7). Define the test function perturbations (the change of scale $\tau/\epsilon^2 \to \tau$ yielding the right sides of the equations below will be used frequently and often without specific mention)

$$f_0^\epsilon(x,t) = \int_t^\infty E_t^\epsilon f_x'(x) G_0(x,\xi^\epsilon(\tau))d\tau = \epsilon^2 \int_{t/\epsilon^2}^\infty E_t^\epsilon f_x'(x)G_0(x,\xi(\tau))d\tau = O(\epsilon^2)|\xi^\epsilon(t)|,$$

$$f_1^\epsilon(x,t) = \int_t^\infty E_t^\epsilon f_x'(x)F(x,\xi^\epsilon(\tau))d\tau/\epsilon = \epsilon \int_{t/\epsilon^2}^\infty E_t^\epsilon f_x'(x)F(x,\xi(\tau))d\tau = O(\epsilon)|\xi^\epsilon(t)|,$$

$$f_2^\epsilon(x,t) = \frac{1}{\epsilon^2}\int_t^\infty d\tau \int_\tau^\infty ds\Big\{E_t^\epsilon[f_x'(x)F(x,\xi^\epsilon(s))]_x' F(x,\xi^\epsilon(\tau))$$

$$- E[f_x'(x)F(x,\xi^\epsilon(s))]_x' F(x,\xi^\epsilon(\tau))\Big\}$$

$$= \epsilon^2 \int_{t/\epsilon^2}^\infty d\tau \int_\tau^\infty ds\Big\{E_t^\epsilon[f_x'(x)F(x,\xi(s))]_x' F(x,\xi(\tau))$$

$$- E[f_x'(x)F(x,\xi(s))]_x' F(x,\xi(\tau))\Big\} = O(\epsilon^2)[|\xi^\epsilon(t)|^2 + 1].$$

The $|\xi^\epsilon(t)|$ terms come from the Gauss-Markov property.

Define

$$f^\epsilon(t) = f(x^\epsilon(t)) + \sum_{i=0}^{2} f_i^\epsilon(x^\epsilon(t),t).$$

The operator $\hat{A}^{m^\epsilon,\epsilon}$ and its domain $\mathcal{D}(\hat{A}^{m^\epsilon,\epsilon})$ is defined in the Appendix. By a direct calculation, using the correlation and conditional expectation properties of the Gauss-Markov process $\xi(\cdot)$, we get that $f(x^\epsilon(\cdot))$ and the $f_i^\epsilon(x^\epsilon(\cdot),\cdot)$ are all in $\mathcal{D}(\hat{A}^{m,\epsilon})$, and

$$\hat{A}^{m^\epsilon,\epsilon} f(x^\epsilon(t)) = f_x'(x^\epsilon(t))\dot{x}^\epsilon(t)$$

$$\hat{A}^{m^\epsilon,\epsilon} f_0^\epsilon(x^\epsilon(t),t) = -f_x'(x^\epsilon(t))G_0(x^\epsilon(t),\xi^\epsilon(t))$$

$$+ \int_t^\infty [E_t^\epsilon f_x'(x^\epsilon(t))G_0(x^\epsilon(t),\xi^\epsilon(s))]_x' \dot{x}^\epsilon(t)\, ds/\epsilon$$

$$\hat{A}^{m^\epsilon,\epsilon} f_1^\epsilon(x^\epsilon(t),t) = -f_x'(x^\epsilon(t))F(x^\epsilon(t),\xi^\epsilon(t))/\epsilon$$

$$+ \int_t^\infty ds\ [E_t^\epsilon f_x'(x^\epsilon(t))F(x^\epsilon(t),\xi^\epsilon(s))]_x' \dot{x}^\epsilon(t)/\epsilon,$$

and similarly for $\hat{A}^{m^\epsilon,\epsilon} f_2^\epsilon(x^\epsilon(t),t)$. See the very similar calculation in [7, Chapter 4] or in [15] where the dynamical terms depend smoothly on x, and are right continuous in t.

We have

$$\text{(2.3a)} \qquad |f(x^\epsilon(t)) - f^\epsilon(t)| = O(\epsilon)[|\xi^\epsilon(t)|^2 + 1].$$

By adding the $\hat{A}^{m^\epsilon,\epsilon} f_i^\epsilon(t)$ to $\hat{A}^{m^\epsilon,\epsilon} f(x^\epsilon(t))$, subtracting from $A^{m^\epsilon} f(x^\epsilon(t))$ and cancelling terms where possible we get

$$\text{(2.3b)} \qquad |\hat{A}^{m^\epsilon,\epsilon} f^\epsilon(t) - A^{m^\epsilon} f(x^\epsilon(t))| = O(\epsilon)[|\xi^\epsilon(t)|^3 + 1].$$

All the $O(\epsilon)$ are uniform in t, ϵ, ω. By equation 4 of the Appendix (with our f^ϵ replacing the q there), the function

$$\text{(2.4)} \qquad M_f^\epsilon(t) = f^\epsilon(t) - f^\epsilon(0) - \int_0^t \hat{A}^{m^\epsilon,\epsilon} f^\epsilon(s)ds$$

is a zero mean martingale. We next show that $M_f^\epsilon(t)/t \xrightarrow{P} 0$ as $t \to \infty$ and $\epsilon \to 0$ in any way at all.

Write (where [t] denotes the greatest integer part of t)

$$\text{(2.5)} \qquad \frac{M_f^\epsilon(t)}{t} = \frac{1}{t}[(M_f^\epsilon(t) - M_f^\epsilon([t])) + M_f^\epsilon(0)] + \frac{1}{t}\sum_{n=0}^{[t]-1} [M_f^\epsilon(n+1) - M_f^\epsilon(n)].$$

Using the fact that $f(\cdot)$ is bounded and (2.3), (2.5) and the martingale property of $M_f^\epsilon(\cdot)$, we get that $E[M_f^\epsilon(t)/t]^2 = O(1)/t$. The fact that M_t^ϵ/t, $f^\epsilon(t)/t$ and $f^\epsilon(0)/t$ all go to zero in probability as $t \to \infty$ (uniformly in ϵ)

together with (2.4) and the second line of (2.3) implies that as $t \to \infty$ and $\epsilon \to 0$,

$$\text{(2.6a)} \qquad \int_0^t A^{m^\epsilon} f(x^\epsilon(s))ds/t \xrightarrow{P} 0.$$

By the definition of $P_T^{m^\epsilon,\epsilon}(\cdot)$, (2.6a) can be written as

$$\text{(2.6b)} \qquad \int A^\alpha f(x) P_T^{m^\epsilon,\epsilon}(dxd\alpha) \xrightarrow{P} 0, \text{ as } T \to \infty \text{ and } \epsilon \to 0.$$

Now, let the control be the classical control function $u^\delta(\cdot)$, and choose a weakly convergent subsequence of the set of random variables $\{P_T^{u^\delta,\epsilon}(\cdot), \epsilon, T\}$ (and also that $\frac{1}{t}\int_0^t A^{u^\delta} f(x^\epsilon(s))ds \to 0$ w.p.1 for all $f(\cdot) \in \mathcal{F}$), indexed by ϵ_n, T_n, and with (random) limit denoted by $\bar{\mu}(\cdot)$. We let the limits $\bar{\mu}(\cdot)$ be defined on some probability space $(\bar{\Omega}, \bar{P}, \bar{\mathcal{F}})$ with generic variable $\bar{\omega}$. Now, (2.6b) implies that

$$\text{(2.7)} \qquad \int A^{u^\delta} f(x)\bar{\mu}(dx) = 0, \quad \bar{P}\text{-almost all } \bar{\omega}.$$

Since our class of $f(\cdot)$ is measure determining, (2.7) implies that almost all realizations of $\bar{\mu}(\cdot)$ are invariant measures for (1.1) (under u^δ). [This is proved by a slight extension of Prop. 9.2 of [8].] By uniqueness of the invariant measure, we can take $\mu(u^\delta,\cdot) = \bar{\mu}(\cdot)$ for all $\bar{\omega}$, and the limit $\bar{\mu}(\cdot)$ does not depend on the chosen subsequence ϵ_n, T_n. Furthermore, by the definition of $P_T^{u^\delta,\epsilon}(\cdot)$,

$$\int_0^t k(x^\epsilon(s),u^\delta(x^\epsilon(s))ds/t = \int_0^t k(x,u^\delta(x))P_t^{u^\delta,\epsilon}(dx)$$
$$\xrightarrow{P} \int k(x,u^\delta(x))\mu(u^\delta,dx) = \bar{\gamma}(u^\delta).$$

Next, choose a weakly convergent subsequence of $\{P_T^{m^\epsilon,\epsilon}(\cdot),\epsilon,T\}$ (and also such that (2.6a) $\to 0$ w.p.1 for all $f(\cdot) \in \mathcal{F}$) indexed by ϵ_n, T_n, and with limit denoted by $\tilde{P}(\cdot)$ (again, defined on some probability space $(\bar{\Omega}, \bar{P}, \bar{\mathcal{F}})$). For each $\bar{\omega}$, we can factor $\tilde{P}(\cdot)$ as $\tilde{P}(dxd\alpha) = m_x(d\alpha)\mu(dx)$. We can suppose that the $m_x(B)$ are x-measurable for each Borel B and $\bar{\omega}$.

By (2.6), for all $f(\cdot) \in \mathcal{F}$,

$$\int A^{\alpha} f(x) m_x(d\alpha)\mu(dx) = 0 \quad \text{for } \bar{P}\text{-almost all } \bar{\omega}. \tag{2.8}$$

This implies that (for a.a.$\bar{\omega}$), $\mu(\cdot)$ is an invariant measure for the process (1.1) with relaxed feedback control $m_x(\cdot)$. As above we also have

$$\int k(x,\alpha) m_x(d\alpha)\mu(dx) = \lim_{\epsilon_n,T_n} \gamma_{T_n}^{\epsilon_n}(m^\epsilon) = \bar{\gamma}(m_x). \tag{2.9}$$

But, by the δ-optimality of $u^\delta(\cdot)$, for almost all $\bar{\omega}$ we have $\bar{\gamma}(m_x) \ngtr \bar{\gamma}(u^\delta) - \delta$. Since this is true for all the limits of the tight set $\{P_T^{m^\epsilon,\epsilon}(\cdot);\epsilon,T\}$, (1.8b) follows. Q.E.D.

3. Alternative Conditions

In this section we redo Theorem 1 under somewhat different conditions. The perturbed test function is only 'first order' here and (2.3) won't hold. But similar results are obtained via a direct averaging method of the type introduced in [5, Chapter 5]. We will use either bounded 'mixing' or Gaussian noise, as in Section 2, and subsets of the following conditions. Let E_t denote the expectation given $\xi(s)$, $s \leq t$.

A3.1. $\xi(\cdot)$ *is bounded, and right continuous* $G_0(\cdot,\cdot)$, $\bar{G}(\cdot,\cdot)$, $F(\cdot,\cdot)$, $F_x(\cdot,\cdot)$ *are continuous.*

A3.2.
$$\int_t^\infty E_t F(x,\xi(s))ds,$$

$$\int_t^\infty E_t[f_x'(x)F(x,\xi(s))]_x' F(x,\xi(t))ds$$

are bounded and x-*continuous uniformly on each compact* x-*set and uniformly in* t, ω.

A3.3.
$$\frac{1}{T}\int_t^{t+T} E_t G_0(x,\xi(s))ds \xrightarrow{P} 0,$$

for each x *as* t *and* $T \to \infty$.

A3.4. *There are continuous* $\bar{F}(\cdot)$, $a(\cdot)$ *such that with* A_0 *given by*

$$A_0 f(x) = f_x'\bar{F}(x) + \frac{1}{2}\sum_{i,j} a_{ij}(x) f_{x_i x_j}(x),$$

we have

$$\frac{1}{T}\int_t^{t+T} ds \int_s^\infty du\ E_t[f_x'(x)F(x,\xi(u))]_x' F(x,\xi(s)) \xrightarrow{P} A_0 f(x),$$

for each x *as* t *and* $T \to \infty$.

A3.5. $\xi(\cdot)$ <u>is a stable Gauss-Markov process, with a stationary transition function.</u> <u>and</u> $F(x,\xi) = F(x)\xi$, $G_0(x,\xi) = G_0(x)\xi$, <u>and</u> $F(\cdot)$, $\bar{G}(\cdot,\cdot)$ <u>and</u> $G_0(\cdot)$ <u>have the smoothness of</u> (A3.1). [We continue to define $\bar{F}(\cdot)$, $a(\cdot)$ and A_0 as in (A3.4), when (A3.5) is used.]

As in Section 1, set $A^{\alpha}f(x) = f_x'(x)\bar{G}(x,\alpha) + A_0 f(x)$, and $\bar{b}(x,\alpha) = \bar{G}(x,\alpha) + \bar{F}(x)$.

Theorem 2. <u>Assume</u> (A2.6) <u>to</u> (A2.9) <u>and either</u> (A3.1) <u>to</u> (A3.4) <u>or else</u> (A3.5). <u>Then</u> (1.8a) <u>and</u> (1.8b) <u>hold</u>.

Proof. Let $f(\cdot)$ be as in Theorem 1. We use the 'direct averaging first order perturbed test function method' of [5, Chapter 5], [9], [1], but the development here is self contained. Define $f_1^{\epsilon}(x,t)$ as in Theorem 1 and set $f^{\epsilon}(t) = f(x^{\epsilon}(t)) + f_1^{\epsilon}(x^{\epsilon}(t),t)$. *Then, (write x for $x^{\epsilon}(t)$ for convenience here)* $f^{\epsilon}(\cdot) \in \mathcal{D}(\hat{A}^{m^{\epsilon},\epsilon})$ and

$$\hat{A}^{m^{\epsilon},\epsilon} f^{\epsilon}(t) = f_x'(x)\Big[\int \bar{G}(x,\alpha)m_t^{\epsilon}(d\alpha) + G_0(x,\xi^{\epsilon}(t))\Big]$$

$$+ \frac{1}{\epsilon^2}\int_t^{\infty} ds[E_t^{\epsilon} f_x'(x)F(x,\xi^{\epsilon}(s))]_x' \, F(x,\xi^{\epsilon}(t))$$

$$+ \text{ terms of order } O(\epsilon)[|\xi^{\epsilon}(t)|^2 + 1].$$

(See the expressions given above (2.3).) Using the scale change $s/\epsilon^2 \to s$, the second term can be seen to be bounded in mean square for the bounded noise case and $O(1)[|\xi^{\epsilon}(t)|^2 + 1]$ in the Gaussian case.

Define the martingale

$$M_f^\epsilon(t) = f^\epsilon(t) - f^\epsilon(0) - \int_0^t \hat{A}^{m^\epsilon,\epsilon} f^\epsilon(s)ds.$$

If

$$(3.1) \qquad M_f^\epsilon(t)/t \xrightarrow{P} 0 \text{ as } \epsilon \to 0,\ t \to \infty,$$

then as in Theorem 1, we have

$$\int_0^t \hat{A}^{m^\epsilon,\epsilon} f^\epsilon(s)ds/t \xrightarrow{P} 0, \text{ as } \epsilon \to 0,\ t \to \infty.$$

If we also have that

$$(3.2) \qquad \frac{1}{t}\int_0^t \left[\hat{A}^{m^\epsilon,\epsilon} f^\epsilon(s) - A^{m^\epsilon} f(x^\epsilon(s))\right]ds \xrightarrow{P} 0, \text{ as } \epsilon \to 0,\ t \to \infty,$$

(and also for u^δ used in lieu of $m^\epsilon(\cdot)$) then the proof can be completed as in Theorem 1. Thus, we need only show (3.1) and (3.2).

To get (3.1), we use the representation (2.5). The martingale difference $M_f^\epsilon(n+1) - M_f^\epsilon(n)$ equals

$$(3.3) \qquad \begin{aligned} &f^\epsilon(n+1) - f^\epsilon(n) - \int_n^{n+1} ds\left[f_x'(x^\epsilon(s))\int \overline{G}(x^\epsilon(s),\alpha)m_s^\epsilon(d\alpha) + G_0(x^\epsilon(s),\xi^\epsilon(s))\right] \\ &\qquad + \int_n^{n+1} dsO(1)[|\xi^\epsilon(s)|^2 + 1]. \end{aligned}$$

Since the mean square value of (3.3) is bounded uniformly in n, ω, ϵ, we get that $E[M_f^\epsilon(t)]^2/t = O(1/t)$ and (3.1) holds, exactly as for Theorem 1.

We now prove (3.2). To <u>simplify the proof, we drop the terms</u> $\int\overline{G}(x,\alpha)m_t^\epsilon(d\alpha)$ <u>and</u> $G_0(x,\xi)$. The first dropped term causes no problems (as in Theorem 1) and the second is dealt with by an averaging method similar to that employed below. Now, we have

$$\frac{1}{t}\int_0^t \hat{A}^{m^\epsilon,\epsilon} f^\epsilon(s)ds$$

$$= \frac{1}{t}\int_0^t ds \int_s^\infty du\ E_s^\epsilon[f_x'(x^\epsilon(s))F(x^\epsilon(s),\xi^\epsilon(u))]_x' F(x,\xi^\epsilon(s))/\epsilon^2$$

$$(3.4) \qquad + \text{ negligable terms}$$

$$= \frac{\epsilon^2}{t}\int_0^{t/\epsilon^2} ds \int_s^\infty du\ E_s[f_x'(x^\epsilon(\epsilon^2 s)F(x^\epsilon(\epsilon^2 s),\xi(u))]_x' F(x^\epsilon(\epsilon^2 s),\xi(s))$$

$$+ \text{ negligable terms.}$$

where the negligeable terms go to zero in the mean square sense as $\epsilon \to 0$. Henceforth, for simplicity, we consider the scalar case and work with only the term $f_{xx}(x)F(x,\xi(u))F(x,\xi(s))$ in (3.4). Write $t = N\Delta$ for integer N and $\Delta > 0$. Define

$$Q^\epsilon(x,s) = \int_s^\infty du\ E_s f_{xx}(x)F(x,\xi(u))F(x,\xi(s)).$$

Then the desired term in (3.4) can be written as

$$(3.5) \qquad \begin{aligned} &\frac{1}{N}\sum_1^N \frac{\epsilon^2}{\Delta} \int_{i\Delta/\epsilon^2}^{(i\Delta+\Delta)/\epsilon^2} ds\ [E_{i\Delta}^\epsilon Q^\epsilon(x^\epsilon(\epsilon^2 s),s) - Q^\epsilon(x^\epsilon(\epsilon^2 s),s)] \\ &+ \frac{1}{N}\sum_1^N \frac{\epsilon^2}{\Delta} \int_{i\Delta/\epsilon^2}^{(i\Delta+\Delta)/\epsilon^2} E_{i\Delta}^\epsilon Q^\epsilon(x^\epsilon(\epsilon^2 s),s)ds. \end{aligned}$$

Since $E|E_{i\Delta}^\epsilon Q^\epsilon(x^\epsilon(\epsilon^2 s),s) - Q^\epsilon(x^\epsilon(\epsilon^2 s),s)|^2$ is bounded uniformly in s, ϵ and Δ, the first set of summands in (3.5) are martingale differences with uniformly (in ϵ, N, t) bounded mean square values. Thus the first sum is $O(1/N)$ and goes to zero in probability as $N \to \infty$, uniformly in ϵ, t. By [5, Chapter 3, Theorem 4, Part 1], and the uniform integrability of $\{\hat{A}^{m^\epsilon,\epsilon} f^\epsilon(t), \epsilon > 0, t < \infty\}$, the sequence $\{x^\epsilon(i\Delta+\cdot) - x^\epsilon(i\Delta), i, \Delta > 0, \epsilon > 0\}$ is tight in $D[0,\infty)$

(Skorohod topology). Because of this, we can replace the $x^\epsilon(\epsilon^2 s)$ in the ith summand of the second term in (3.5) by $x^\epsilon(i\Delta)$ for all i, and only alter the sum by an amount which goes to zero in probability (uniformly in ϵ and N) as $\Delta \to 0$.

Doing this replacement and using either the Gaussian property (A3.5) or else (A3.4) for the bounded noise case, and the continuity of $F(\cdot,\xi)$ (uniform in ξ in the bounded noise case) and the continuity and compact support of $f_{xx}(\cdot)$ yields that the second sum in (3.5) and

$$\frac{1}{N}\sum_{1}^{N}\frac{\epsilon^2}{\Delta}\int_{i\Delta/\epsilon^2}^{(i\Delta+\Delta)/\epsilon^2} ds\, f_{xx}(x^\epsilon(i\Delta))a(x^\epsilon(i\Delta)) \tag{3.6}$$

have the same limit in probability as $N \to \infty$, $\Delta \to 0$, $\epsilon \to 0$, $N\Delta \to \infty$. We next use the tightness of $\{x^\epsilon(i\Delta + \cdot) - x^\epsilon(i\Delta),\ i,\ \Delta > 0,\ \epsilon > 0\}$ again to replace the $x^\epsilon(i\Delta)$ in (3.6) by $x^\epsilon(\epsilon^2 s)$, and get the same result; namely that the limit in probability is the same as $N \to \infty$, $\Delta \to 0$, $\epsilon \to 0$, $N\Delta \to \infty$. Finally, repeating the procedure approximation procedure used from (3.5) on for the various neglected terms yields (3.2). Q.E.D.

4. Extensions

Discrete time problem. There are direct extensions to the discrete parameter model

(4.1) $$X^{\epsilon}_{n+1} = X^{\epsilon}_{n} + \epsilon \int \bar{G}(X^{\epsilon}_{n},\alpha)m_{n}(d\alpha) + \epsilon G_{0}(X^{\epsilon}_{n},\xi^{\epsilon}_{n}) + \sqrt{\epsilon}\, F(X^{\epsilon}_{n},\xi^{\epsilon}_{n}).$$

In both (4.1) and (2.1), we can allow some 'state dependence' of the noise -- (cf, the 'Markov' dependent type used in [5, Chapters 4.4 or 5.5].)

Approximate non-linear filtering. In the following two applications, there is no control. In Section 7 of [10], an 'approximate' non-linear filtering problem was dealt with, where the system driving and observation noises were wideband. It was shown (under a condition concerning the uniqueness of a certain invariant measure) that the average error (using the notation of that paper)

(4.1) $$\lim_{\epsilon} \frac{1}{T} \int_{0}^{T} E[\phi(x^{\epsilon}(t)) - (P^{\epsilon}(t),\phi)]^{2}dt$$

converged to what one would get if the true optimal filter were used on the 'limit' process. Here $x^{\epsilon}(\cdot)$ is the state of the 'signal system' (say, of the form (2.1)), $\phi(\cdot)$ is bounded and continuous, and $P^{\epsilon}(\cdot)$ is the measure valued output (not necessarily the conditional distribution) of the 'approximate' filters used in [12]. Via the technique of this paper, similar results can be obtained if the E in (4.1) were dropped. This is useful, since we would normally filter only one path -- over a long time -- and the use of the expectation might give an inappropriate measure of the filter performance.

Liapunov exponents for wide bandwidth noise driven systems. The theory of Liapunov exponents is well developed for systems of the form

$$dx = Ax\,dt + \sum_{i=1}^{k} B_i x \circ dw_i, \tag{4.2}$$

where the 'o' denotes that the stochastic integral is in the 'Stratonovich' sense and where the $w_i(\cdot)$ are real valued and mutually independent standard Wiener processes [11]. The 'Stratonovich' sense integral is used to be consistent with the usage in [11] and because it simplifies the identification of the limit process and its 'projection' below in this case. Of practical interest are the convergence properties of numerical methods of evaluating these exponents, as well as the study of the asymptotic behavior of wideband noise driven systems

$$\dot{x}^{\epsilon} = Ax^{\epsilon} + \sum_{i=1}^{k} B_i\, x\xi_i^{\epsilon}, \tag{4.3}$$

via the method of Liapunov exponents. In (4.3), the $\xi_i^{\epsilon}(\cdot)$ are orthogonal and scalar valued processes. Of particular interest is whether the exponents for (4.3) converge to those for the limit system (which will be of the general form of (4.2)) as $\epsilon \to 0$.

Under the conditions of Theorem 2 on $\xi_i^{\epsilon}(\cdot) = \xi(\cdot/\epsilon^2)$, the above orthogonality condition, and the normalization

$$\frac{1}{T}\int_t^{t+T} ds \int_s^{\infty} E_t \xi_i(s)\xi_j(u)du \to \frac{1}{2}$$

in probability as t and T go to ∞, the $x(\cdot)$ of (4.2) is the weak limit of (4.3), if the initial conditions converge. We can assume this normalization to hold in general, since otherwise we absorb the 'constants' into the B_i in the obvious way.

Define $y = x/|x|$. Then

$$\dot{y} = \dot{x}/|x| - x[x'\dot{x}]/|x|^{3/2}$$

and

$$\dot{y}^\epsilon = Ay^\epsilon + \sum_{i=1}^{k} B_i y^\epsilon \xi_i^\epsilon - y[y'Ay] - y\left[y' \sum_{i=1}^{k} \xi_i^\epsilon B_i y\right]. \tag{4.4}$$

Assume the noise conditions of Theorem 2. Then, it is not hard to show that $P\{x^\epsilon(s) \neq 0, \text{ any } s \leq T\} = 1$ for all ϵ, T.

Of interest is the calculation of quantities such as $\lim_t E\int_0^t q(y^\epsilon(s))ds/t$ for bounded and continuous $q(\cdot)$. In the Monte Carlo evaluation of the limit, one often uses,

$$\frac{1}{t}\int_0^t q(y^\epsilon(s))ds \tag{4.5}$$

for large t and some small ϵ, and it is of interest to know whether or not the convergence is to the correct limit and whether it is uniform in ϵ and t in the sense of (1.8a). [An alternative is of course to fix $T < \infty$ and approximate $E\int_0^T q(y^\epsilon(s))ds/T$ for small ϵ by taking many independent runs and averaging. But, the 'uniformity' questions still arise.]

Define $y(t) = x(t)/|x(t)|$ and

$$q(y) = y'Ay + \frac{1}{2}\sum_{i=1}^{k} [y'(B_i + B_i')B_i y - (y'B_i y)^2],$$

and assume that $y(\cdot)$ has a unique invariant measure on the sphere (this is true under a Lie algebraic condition on the set $(A, B_i, i \leq k)$ [11]). Then [11] the (maximal) Liapunov exponent is the limit (which is a constant w.p.1)

$$\lim_t \int_0^t q(y(s))ds/t. \tag{4.6}$$

One is interested in whether (4.5) converges to (4.6) as $\epsilon \to 0$ and $t \to \infty$.

By Theorem 2 $(x^\epsilon(\cdot),y^\epsilon(\cdot)) \Rightarrow (x(\cdot),y(\cdot))$ (Skorohod topology), and the weak limit process $y(\cdot)$ is characterized completely by the correlation functions of the $\xi_i(\cdot)$. Let $\mu(\cdot)$ denote the assumed unique invariant measure for $y(\cdot)$. Then

$$\frac{1}{t}\int_0^t q(y^\epsilon(s))ds \xrightarrow{P} \int q(y)\mu(dy), \quad \text{as} \quad \epsilon \to 0, \quad t \to \infty, \tag{4.7}$$

and the limit value is just the (maximum) Liapunov exponent for $x(\cdot)$. The general method is applicable to a wide variety of noise processes and can readily be extended to yield convergence of various numerical approximations to the (maximal) Liapunov exponent for (4.2), via use of either a discrete time approximation to (4.2) or the various interpolations which can be used to approximate the stochastic integrals.

Appendix

Definition. Let U be a compact set in some Euclidean space. Let the $w(\cdot)$ in (1.1) be a Wiener process with respect to a filtration $\{\mathcal{F}_t\}$. A measure valued (a measure on the Borel sets of $U \times [0,\infty)$) random variable $m(\cdot)$ is an admissible relaxed control if $\int\int_0^t f(s,\alpha)m(dsd\alpha)$ is progressively measurable for each bounded and continuous $f(\cdot)$ and $m([0,t] \times U) = t$. If $m(\cdot)$ is admissible, then there is a derivative $m_t(\cdot)$ (defined for almost all t) which is non-anticipative and

$$\int_0^t\int f(s,\alpha)m(dsd\alpha) = \int_0^t ds \int f(s,\alpha)m_s(d\alpha)$$

for all t w.p.1. Sometimes we use the 'feedback' relaxed control (which we write as $m_x(\cdot)$) which is a measure on the Borel sets of U for each x and $m_x(B)$ is Borel-measurable for each Borel B. The $m_t(\cdot)$ and $m_x(\cdot)$ will also be referred to as relaxed controls.

An admissible relaxed contol $m(\cdot)$ for (2.1) is also a measure valued random variable (as above) but $\int_0^t f(s,\alpha)m(dsd\alpha)$ is progressively measurable with respect to $\{\mathcal{F}_t^\epsilon\}$, where $\mathcal{F}_t^\epsilon$ is the minimal σ-algebra measuring $\{\xi^\epsilon(s), x^\epsilon(s), s \leq t\}$. Also, we impose $m([0,t] \times U) = t$. As above, there is also a derivative $m_t(\cdot)$, where the $m_t(B)$ are $\mathcal{F}_t^\epsilon$ measurable for Borel B. We sometimes use the symbol $m^\epsilon(\cdot)$ or $m_t^\epsilon(\cdot)$ for the relaxed controls, when (2.1) is used.

Definition. Let $q(\cdot)$ be progressively measurable with respect to $\{\mathcal{F}_t^\epsilon\}$. Suppose that there is a progressively measurable (with respect to $\{\mathcal{F}_t^\epsilon\}$) $g(\cdot)$

such that

$$\text{(1)}\qquad \sup_{t\leq T} E|g(t)| < \infty,\quad E|g(t+s) - g(t)| \to 0 \text{ as } s \downarrow 0, \text{ each } t,$$

$$\text{(2)}\qquad \sup_{\substack{t\leq T\\ \delta>0}} E\left|\frac{E_t^\epsilon q(t+\delta) - q(t)}{\delta} - g(t)\right| < \infty$$

$$\text{(3)}\qquad \lim_{\delta\downarrow 0} E\left|\frac{E_t^\epsilon q(t+\delta) - q(t)}{\delta} - g(t)\right| \to 0, \text{ each } t.$$

Then we say that $q(\cdot) \in \mathcal{D}(\hat{A}^{m,\epsilon})$, the domain of the operator $\hat{A}^{m,\epsilon}$ and that $\hat{A}^{m,\epsilon}q = g$. If $q(\cdot) \in \mathcal{D}(\hat{A}^{m,\epsilon})$, then [3, Chapter 3], [12],

$$\text{(4)}\qquad q(t) - \int_0^t \hat{A}^{m,\epsilon} q(s)ds$$

is a martingale. This martingale property will be heavily used in the proofs. We define $\hat{A}^{\alpha,\epsilon}$ to be $\hat{A}^{m,\epsilon}$ with m_t concentrated at α, and $\hat{A}^{u,\epsilon}$ is defined in the obvious way.

The form given for $\hat{A}^{m,\epsilon}$ in Theorem 1 satisfy (1) - (3) if $\int \bar{G}(x,\alpha)m_t(d\alpha)$ is right continuous w.p.1. Generally, since we are only concerned with the use of $\hat{A}^{m,\epsilon}q$ in an integral – to get the martingale property (4) – the given forms work in general. Alternatively, they are precisely what one gets via the following procedure. Let $t = N\Delta$ for integer N and consider the following expression for $\{\mathcal{F}_t^\epsilon\}$-progressively measurable $q(\cdot)$ with $\sup_t E|q(t)| < \infty$

$$\text{(5)}\qquad q(t) - q(0) - \sum_{i=0}^{N-1} E_{i\Delta}^\epsilon[q(i\Delta + \Delta) - q(i\Delta)].$$

Suppose that there is a progressively measurable $g(\cdot)$ such that the right side of (5) converges to $\int_0^t g(s)ds$ in mean as $\Delta \to 0$, for each t. Then

$$q(t) - q(0) - \int_0^t g(s)ds \tag{6}$$

is a zero mean $(\mathcal{F}_t^\epsilon)$-martingale and we write $q \in \mathcal{D}(\hat{A}^{m,\epsilon})$ and $g = \hat{A}^{m,\epsilon}q$.

5. Convergence of Pathwise Discounted Costs to the Ergodic Cost

In this section, we treat the discounted cost result (1.9). Again, the exact sense in which the $m^\epsilon(\cdot)$ are δ_1-optimal is left a little vague. Since $u^\delta(\cdot)$ is asymptotically δ-optimal, no matter what the $m^\epsilon(\cdot)$ are, the pathwise costs are (for small β,ϵ) no better (modulo 2δ) than the costs for the $m^\epsilon(\cdot)$, with an arbitrary large probability.

Theorem 3. Under the conditions of either Theorem 1 or 2, the limits (1.9) hold.

Remarks on the Proof. The proof is essentially the same as those of Theorems 1 or 2, and we only remark on the differences. We use the discounted occupation measures

(5.1)
$$P_\beta^{m,\epsilon}(B\times C) = \beta \int_0^\infty e^{-\beta t} I_{\{x^\epsilon(t)\in B\}} m_t(C)dt,$$
$$P_\beta^{m}(B\times C) = \beta \int_0^\infty e^{-\beta t} I_{\{x(t)\in B\}} m_t(C)dt$$

and analogously for the feedback control cases.

Then the cost can be written as

$$V_\beta^\epsilon(m^\epsilon) = \int k(x,\alpha) P_\beta^{m^\epsilon,\epsilon}(dx\,d\alpha).$$

By the tightness conditions (A2.7), (A2.8), the $\{P_\beta^{m^\epsilon,\epsilon}(\cdot)\}$ and $\{P^{u^\delta,\epsilon}(\cdot)\}$ are tight. Define

$$f_\beta^\epsilon(t) = \beta e^{-\beta t} f^\epsilon(t). \tag{5.2}$$

This will be used in lieu of the $f^{\epsilon}(\cdot)$ in either Theorems 1 or 2. We have

$$\hat{A}^{m^{\epsilon},\epsilon} f^{\epsilon}_{\beta}(t) = -\beta^2 e^{-\beta t} f^{\epsilon}_{\beta}(t) + \beta e^{-\beta t} \hat{A}^{m^{\epsilon},\epsilon} f^{\epsilon}_{\beta}(t). \tag{5.3}$$

Define the martingale

$$\begin{aligned} & f^{\epsilon}_{\beta}(t) - f^{\epsilon}_{\beta}(0) - \int_0^t \hat{A}^{m^{\epsilon},\epsilon} f^{\epsilon}_{\beta}(s)ds \\ & = \beta e^{-\beta t} f^{\epsilon}(t) - \beta f^{\epsilon}(0) - \int_0^t [-\beta^2 e^{-\beta s} f^{\epsilon}(s) + \beta e^{-\beta s} \hat{A}^{m^{\epsilon},\epsilon} f^{\epsilon}(s)]ds. \end{aligned}$$

As in Theorems 1 or 2

$$0 = \lim_{\substack{(\beta,\epsilon)\to 0 \\ t\to\infty}} \beta \int_0^t e^{-\beta s} A^{m^{\epsilon}} f(x^{\epsilon}(s))ds. \tag{5.4}$$

Thus

$$0 = \lim_{(\beta,\epsilon)\to 0} \int A^{\alpha} f(x) P^{m^{\epsilon},\epsilon}_{\beta}(dx d\alpha). \tag{5.5}$$

Again we choose weakly convergent subsequences of the $\{P^{m^{\epsilon},\epsilon}_{\beta}(\cdot)\}$ or $\{P^{u^{\delta},\epsilon}_{\beta}(\cdot)\}$ and continue as in the proofs of either Theorems 1 or 2 to get Theorem 3.

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